Light Field Geometry

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Abstract

We propose a general mathematical framework for dealing with Light Fields:

The Light Field is a 2-form on Light Space (LS). Light Space is the set of all rays in 3D, and is locally isomorphic to the Grassmann manifold G^4_2. The Light Field form is defined as the pull-back of the brightness form of the observed surface. It satisfies equations related to the conservation law in fluid dynamics (or optical flow), and to Maxwell's equations, with images as boundary conditions. Due to the existance of a projection onto the viewed surface, LS is a bundle, and the camera itself is a section on it. The image is produced as a restriction of the Light Field to that section.

Practical applications that we are targeting are Light Field reconstructions and Image Based 3D, in general.

1 Introduction

The Light Field [1] is a new, and promising approach to 3D graphics. Besides being one of the main approaches to Image Based Rendering, it is also an intrinsic part of all other Image Based applications.

Still, due to the difficult nature of the problem of doing 3D based on the Light Field, it has found very few practical applications. The difficulties are (1) technical and (2) theoretical.

(1) Authors point to the need of a large number of cameras in order to capture the Light Field, and need of high computer power and gigabytes of disk space in order to process the data.

(2) Only first steps have been made towards a complete mathematical model of the Light Field [6], [2]. The theory is still in its infancy.

Thus, our belief is that building a good mathematical framework for treating Light Fields will not only 'shed light' on (2), but will also improve the methods of handling the technical problems (1) through better understanding. This paper is our first attempt to put together a mathematical model for the Light Field.

All expressions and considerations are local. Currently we do not attempt to capture the global nature of surfaces and Light Fields.

2 Main Concepts

In this section we define new terms and give new geometric interpretation to existing terms. We base our discussion on concepts that can be found in [3], [4], [5].

2.1 Brightness

We consider Light Fields created by a Lambertian surface Σ in 3D. The surface reflects light equally in all directions. As a result, each point on the surface defines a pencil of light rays with equal intensity.

Brightness is energy density on the 2D surface Σ. In other words, brightness has to be described by the volume form in 2D. In this way, brightness is a 2-form $\tilde{F}$ defined on Σ. We can integrate over any patch S on Σ to get the total energy $\int_S \tilde{F}$ radiated from it. Note that this is the coordinate free expression for the integral of a 2-form over the manofold Σ on which it is defined.

Locally, the surface can be parametrized by x^1, x^2 as $(x^1, x^2, f(x^1, x^2)) \in R^3$, where $f(x^1, x^2)$ is a function, and then the brightness would be $\tilde{F} = \tilde{F}(x^1, x^2) dx^1 \wedge dx^2$.

2.2 Light Space

Light Space (LS) is the space of all light rays in 3D. As a manifold it is equivalent to the tangent bundle to the sphere S^2. Locally, it is equivalent to the Grassmann manifold G^4_2. As it is well-known, G^4_2 can be represented as the Klein Quadric in the space of skew symmetric tensors in R^4. Tensors belonging to the Klein Quadric satisfy the Plucker condition for decomposability

$$T \wedge T = 0 \tag{1}$$

Often, 'Light Sandwich' coordinates are used in Light Space. Light rays are parametrized by a pair of points at which they intersect two fixed parallel planes [1]. Tensors generated from two points are obviously decomposable and satisfy (1).

We also find useful the 'Screw' coordinates [2] in Light Space: If e_μ are basis vectors in R^4,

$$e_{\mu\nu} = e_\mu \wedge e_\nu \tag{2}$$

is the basis for skew symmetric tensors, which can be written as

$$T = a_1 e_{01} + a_2 e_{02} + a_3 e_{03} + b_1 e_{23} + b_2 e_{31} + b_3 e_{12} \quad (3)$$

We will use the notation

$$T = (\vec{a}, \vec{b}). \quad (4)$$

Because of the projective treatment, we can always consider the first vector normalized,

$$\vec{a} \cdot \vec{a} = 1. \quad (5)$$

The Plucker condition in screw coordinates is

$$\vec{a} \cdot \vec{b} = 0. \quad (6)$$

$\vec{a}$ is the direction, $\vec{b}$ is the 'moment' of the line, and we can always write

$$\vec{b} = \vec{r} \times \vec{a} \quad (7)$$

where $\vec{r}$ is any point on the line.

Note also similarity with the Maxwell tensor, with $\vec{a}$ taking the place of the electric field, $\vec{b}$ - of the magnetic field [7].

2.3 Light Field

The Light Field is a skew symmetric tensor field F (aloso known as 2-form) in Light Space. This field satisfies certain equations, derived later in this work. Now we will describe how the 2-form F is generated. The mechanism is similar to that of a fluid in space-time, or optical flow in video (see Appendix).

A light ray through a point on the surface Σ parametrized by x^1, x^2 can be locally represented as

$$(\vec{a}, (x^1, x^2, f(x^1, x^2)) \times \vec{a}) = (\vec{a}, \vec{b}(x^1, x^2)) \quad (8)$$

All rays through that point have same brightness. In this way the points on the surface define a projection p in Light Space, which maps all lines from each pencil to the point on the surface where that pencil starts from. In our local coordinates this projection is $\pi_2(\vec{a}, (x^1, x^2)) = (x^1, x^2) \in \Sigma$.

The above projection is a map that sends points in Light Space belonging to same pencil to the corresponding point on Σ. Because of that projection, Light Space is a bundle. Sometimes we call it 'Light Bundle generated by a surface Σ'.

The above projection mapping $p : G_2^4 \to \Sigma$ defines a pull-back of any differential form on Σ back to Light Space. In the case of the brightness 2-form $\tilde{F}$, we have:

$$F = p^*(\tilde{F}) \quad (9)$$

This 2-form F is the Light Field. In local coordinates $x^\mu, \mu = 1, 2$ on the surface and $y^i, i = 1, 2, 3, 4$ on Light Space,

$$(p^*(\tilde{F}))_{ij}(y) = \frac{\partial p^\mu}{\partial y^i}(y) \frac{\partial p^\nu}{\partial y^j}(y) \tilde{F}_{\mu\nu}(p(y)). \quad (10)$$

2.4 Camera

A camera is a local section on the Light Bundle. In more detail, Light Space is a bundle $p : G_2^4 \to \Sigma$, and any section on it is a camera. A section is a mapping $\sigma : \Sigma \to G_2^4$ that satisfies $p(\sigma(x)) = x$ for any $x \in \Sigma$. This is simply a condition that the camera is not degenerate, i.e. it sees each point $x \in \Sigma$ only once.

A pinhole camera at a point $\vec{r}$ in 3D is a section in Light Space that is also a pencil defined by that point. All rays captured by the pinhole camera are described in screw coordinates as $(\vec{a}, \vec{r} \times \vec{a})$, and parametrized by the unit sphere (because $\vec{a} \in S^2$).

Other examples of cameras are described in a recent paper [6].

A picture is the restriction of the Light Field 2-form F to the graph of the section. It is a density 2-form, not a scalar function.

3 Equations for the Light Field

The Light Field two-form F is decomposable. This is so because it is a pull-back of the volume form on the surface, which is decomposable.

The condition for decomposability

$$F \wedge F = 0 \quad (11)$$

is completely independent from (1). Another coincidence is that the electromagnetic field of arbitrarily moving point charge is described by decomposable Maxwell tensor [7].

Second property of the Light Field is that F is closed. This is so because it's the pull-back of a closed form on Σ:

$$dF = dp^*(\tilde{F}) = p^*(d\tilde{F}) = 0 \quad (12)$$

Last equality comes from $d\tilde{F} = 0$. Again, electromagnetic field is a closed 2-form.

In order to continue, we need a metric tensor in Light Space. One way is to derive it from the metric in R^4, but this metric itself is not defined. We only know the metric in 3D space. Another approach is to use metric, natural in screw theory. Also, we may choose to pull-back the metric from the surface.

Given a metric g in Light Space, we can define a Hodge $*$ operator. By definition, $*$ sends p-form T_1 to $(n-p)$-form $*T_1$ and is such that for any p-form T_2

$$T_2 \wedge *T_1 = < T_2 T_1 >_g \omega_g \tag{13}$$

where ω_g is the metric volume form. Hodge $*$ encodes the metric stricture into the exterior algebra. Having a $*$ operator makes it possible to calculate the following 1-form in Light Space: $J = *d*F$.

Now we are ready to define our formulation of the problem of reconstructing Light Fields:

Given a 1-form J satisfying $d*J = 0$, and boundary conditons, the Light Field can be found by solving

$$dF = 0 \tag{14}$$

$$*d*F = J \tag{15}$$

We should point out that (14) and (15) are identical to the coordinate free expression for Maxwell's equations. We can use methods and ideas from the very well developed area of electromagnetic theory to solve for Light Fields, even if our metric and coordinates are different.

Notice also that the Light Filed is defined by a current J, which is in a way 'the derivative' of a Light Field. We expect that this current is free from the redundancy of light rays that are constant along fibres in the Light Bundle, and it is zero almost everywhere.

When solving the equations we need boundary conditions. These could be the values of F restricted to certain sections in the Light Bundle. As we know, sections are cameras and define pictures. So, we are back to reconstructing the Light Field from pictures. Note, however, that in our case a small number of pictures must be sufficient.

4 Closing remarks

We point the reader to certain similarity with fluids. $*J$ describes the flow of a conserved fluid in the 4-dimensional space of rays (Light Space). This fluid generates the Light Field. Equation (14) states that the Light Field itself is a conserved fluid flowing in Light Space, constrained by equation (15).

As it is well-known, in the case of $J = 0$ we can act on (14) with $*d*$ and on (15) – with d, and add them to derive the Laplace equation. So, at least to some approximation, the Light Field can be reconstructed by solving the Laplace equation. Again, notice that the Laplacian depends on the metric in Light Space.

With some approximation we expect that a Healing Brush - type reconstruction [8] of the Light Field will work with J extracted from a different area in Light Space to reconstruct missing or damaged area. We do not have to go up to second order Poisson equations, because (14), (15) are first order and iterative solutions are expected to converge faster.

References

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5 Appendix

5.1 Geometric model of a fluid

A fluid in space-time R^4 is the following pair: a bundle $p : R^4 \to R^3$ and a 3-form $\tilde{\alpha}$ on the base R^3. The points of the base are the 'molecules' of the fluid, with density $\tilde{\alpha}$.

If we define $\alpha = p^*(\tilde{\alpha})$ to be the pull-back of the density 3-form to R^4, we can show that α is closed $d\alpha = 0$:

$d\alpha = dp^*(\tilde{\alpha}) = p^*(d\tilde{\alpha}) = 0$

This is the well-known conservation law for fluids (or optical flow in video), written in coordinate-free form. It applies to the case when the fluid is described by means of 3-form in R^4 (or 2-form in R^3).

When in R^4 there is a metric g, then we have the volume form ω_g and there exists a uniquely defined vector field Y satisfying $\omega_g(Y, X_1, X_2, X_3) = \alpha(X_1, X_2, X_3)$ for any vector fields X_1, X_2, X_3. Y is a more common description of the flow.