# Towards understanding Bilateral Filters

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**Abstract.** We show that conventional Bilateral filtering is equivalent to normal filtering of a delta function distribution in (higher dimensional) Image Space. This suggests possible generalizations of Bilateral and Trilateral filters.

## 1 Deriving the Bilateral filter expression

Our model replaces the image $f(x, y)$ with a distribution having support on a surface in Image Space ($R^3$ with coordinates $x, y, z$), which surface can be represented as a function on the $x, y$ image plane:

$$z = f(x, y). \tag{1}$$

To be more specific, intuitively we assume the surface has "thickness" proportional to $z$. As a result, the image $f(x, y)$ is replaced by the following distribution in 3D Image Space:

$$\phi(x, y, z) = z\delta(z - f(x, y)). \tag{2}$$

For a simple practical example of how Bilatteral filtering works, let $c(x, y)$ be a 2-dimensional Gaussian kernel. Let $s(z)$ be a Gaussian kernel. Now, a Gaussian filter with the $c$ and $s$ kernels acting on $\phi$ would be:

$$\psi(u, v, w) = \int z\delta(z - f(x, y))c(x - u, y - v)s(z - w)dxdydz. \tag{3}$$

Integration over $z$ gives us

$$\psi(u, v, w) = \int f(x, y)c(x - u, y - v)s(f(x, y) - w)dxdy. \tag{4}$$

Now, we evaluate $\psi$ on the surface (1). Using the notation $\psi(u, v) = \psi(u, v, f(u, v))$,

$$\psi(u, v) = \int f(x, y)c(x - u, y - v)s(f(x, y) - f(u, v))dxdy. \tag{5}$$

This is exactly the conventional expression for Bilateral filtering. We believe this derivation of (5) is a new result. We believe it can be a starting point for understanding and generalizations of Bilateral.

## 2 Log-space intuition

If expression (2) appears to the reader as chosen "ad hoc", he may want to consider a more intuitive expression for the distribution (produces the same result):

$$\phi(x, y, z) = \delta(\ln z - \ln f(x, y)). \tag{6}$$

It is trivial to see that (6) has support on the same surface (1). Now it explicitly accounts for the fact that our perception of light is logarithmic. (Also, the logarithm removes the problem of how to interpret the negative values of $z$ that we encounter when filtering in conventional non-logarithmic Image Space.) Now (3) is replaced with

$$\psi(u, v, w) = \int \delta(\ln z - \ln f(x, y))c(x - u, y - v)s(z - w)dxdydz. \tag{7}$$

The reader has to perform integration in (7) over z (directly or by using the properties of the delta function), which produces exactly (4). After that we proceed as before.

## 3 Trilateral Filter and beyond ...

The reason for evaluating the distribution $\psi(u, v, w)$ on the surface (1) was to create an image from a distribution that is not an image. Since the original image $f(x, y)$ is noisy, intuitively it is hard to believe that the irregular surface that defines it is "the right" surface on which we need to evaluate $\psi(u, v, w)$. "The right surface" or "the true image" – we don't know what that really means, and/or if this true image exists or can be known. But an approach of connections where this surface is the integral surface of a connection in fibred image space appears to be the right approach.

This suggests the idea that as approximation a smoother version of the original surface might be appropriate for evaluating $\psi(u, v, w)$. At each point this might be a linear interpolation of the original image, which brings us to the Trilateral filter ideas.

Other choices might be some blurred version of $f(x, y)$, and probably evaluation of $\psi(u, v, w)$ on a Bilaterally filtered original. This can be iterated. Which leads us to more questions:

- Which approach is best?
- Is there significant practical benefit of going beyond one iteration?
- Is there any math that would reduce all those iterations to one step?

The reader should not look for a final answer in this section. We are simply trying to find a direction and asking questions. There is only one thing that stands out as more clear: The fibred space approach appears to be just the right framework for Bilateral.